# Lyapunov exponents and transport in the Zhang model of self-organized criticality

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We discuss the role played by Lyapunov exponents in the dynamics of Zhang's model of self-organized criticality. We show that a large part of the spectrum (the slowest modes) is associated with energy transport in the lattice. In particular, we give bounds on the first negative Lyapunov exponent in terms of the energy flux dissipated at the boundaries per unit of time. We then establish an explicit formula for the transport modes that appear as diffusion modes in a landscape where the metric is given by the density of active sites. We use a finite size scaling ansatz for the Lyapunov spectrum, and relate the scaling exponent to the scaling of quantities such as avalanche size, duration, density of active sites, etc.

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## I. INTRODUCTION

Within the past ten years the notion of self-organized criticality (SOC) has become a new paradigm for the explanation of a huge variety of phenomena in nature and social sciences. Its origin lies in the attempt to explain the widespread appearance of power-law-like statistics for characteristic events in a multitude of examples, such as the distribution of the size of earthquakes, $1/f$ noise, amplitudes of solar flares, and species extinction, to name only a few cases [1–3]. In this paradigm, the dynamics occur as chain reactions or avalanches. Furthermore, a stationary regime is reached, where the average incoming flux of external perturbations is balanced by the average outgoing flux that can leave the system at the boundary or by dissipation in the bulk, and there is a constant flux through the system. In this stationary state, referred to as the *SOC state*, the distribution of avalanches follows a power law—that is, there is a scale invariance reminiscent of thermodynamic systems at the critical point. A local perturbation can induce effects at any scale, and there are long-range spatial and time correlations. In other words, in this paradigm the system *spontaneously* reaches a critical state without any fine tuning of some control parameter.

Several models have been proposed to mimic this mechanism, including the sandpile model [1], the abelian sandpile [4] or the continuous energy model [5]. The results available are mainly numerical, and only a few rigorous results are known. Numerical simulations report the following behavior. Fix an observable, say $x$, measuring some property of an avalanche (duration, size, etc.), and compute the related probability $P_L(x)$ at stationarity for a system of characteristic size $L$. The graph of $P_L(x)$ exhibits a power law behavior over a finite range, with a cutoff corresponding to finite size effects. As $L$ increases the power law range increases, leading to the conjecture that a critical state is indeed achieved in the thermodynamic limit, namely, that $P_L(x)$ behaves like $1/x^{\tau_x}$ as $L\to\infty$. $\tau_x$ is called the *critical exponent* for the observable $x$. There is apparently no control parameter to tune in order to attain the critical state. Despite the large number of papers written on the subject, some basic problems are still open.

Guided by the wisdom coming from renormalization group analysis and phase transitions in equilibrium systems, it seems natural to look for a possible classification of the models into universality classes characterized by a set of critical exponents, for a family of ''relevant'' avalanche observables. However, the link between the ''criticality'' of the ''out of equilibrium'' SOC models and the usual statistical mechanics of phase transitions in equilibrium systems remains to be clarified [6]. Furthermore, apart from the fact that the commonly studied observables (size, duration, area, and gyration radius) do not necessarily consitute a *complete* set allowing one to classify the models, even a computation of the critical exponents $\tau_x$ from numerical data is not easy, and there is not yet any agreement on the way to do this. It is clear that the simple measurement of the slope of $P_L(x)$ in the linear range of a log-log plot is not reliable, due to the finite sample fluctuations, and because the explicit form of the cutoff is not known in general. The computation of $\tau_x$ from the behavior of the moments is certainly a better way to proceed. However there is no agreement yet about whether one should use a finite size scaling treatment [7] or more sophisticated methods (such as multifractal analysis [8]). Therefore, at the moment, the identification of a (supposed) universality class seems problematic. Finally, the central question is the following: what exactly does a knowledge of the critical exponents $\tau_x$ teach us about the model?

An alternative approach to better understand the behavior of SOC models can consist of studying the microscopic dynamics and inferring information about the macroscopic behavior from this analysis. A detailed analysis can, at first sight, seem useless since the conventional wisdom from classical statistical mechanics is that microscopic ''details'' are irrelevant, and only structural properties like conservation laws and symmetries are essential. However, as mentioned above, the theory of SOC has not yet reached the level of understanding of classical critical phenomena. It suffers, in particular, from the lack of a thermodynamic formalism, and notions like Gibbs measures and free energy can *a priori* not

 

be used. On the other hand, by having a precise description of the dynamics of the finite size system, one can expect a better understanding of the thermodynamic limit, and can decide which components in the models definition are really ''relevant'' and what information the usually computed quantities (like critical exponents) actually give us.

This is the essence of the program we developed in Refs. [9–11]. We found that Zhang's model of SOC [5] can be fruitfully studied with the tools of hyperbolic dynamical system theory. Then we were able to extract unexpected results, establishing, in particular, a formula relating the critical exponent of avalanche size to the spectrum of the Lyapunov exponents. In this paper we develop this point of view, and make a further step toward understanding the dynamical properties of this model and their link to the SOC state. We first define the model as a hyperbolic dynamical system of skew-product type. We then define two different time scales in this setting: the *local time*, which is the natural time for the dynamical system, and the *avalanche time*, related to the avalanche duration. We introduce a natural invariant measure to characterize the statistical properties at stationarity, and we relate the avalanche observable statistics to the ergodic local time average. We then discuss the role played by Lyapunov exponents in the dynamics, and their relation to the energy transport and the average avalanche observables. We show that random excitation induces a positive Lyapunov exponent, while the relaxation dynamics corresponds to negative exponents. Furthermore, we show that a wide part of the spectrum (slowest modes) is associated with energy transport in the lattice. In particular, we give bounds on the first negative Lyapunov exponents in terms of the energy flux dissipated at the boundaries per unit of time. We establish an explicit formula for the transport modes, these appear as diffusion modes in a landscape where the metric is given by the density of active sites. Except for the first modes, they differ dramatically from the normal diffusion modes that one would obtain by assuming a uniform density of active sites. It was argued in Ref. [12] that SOC requires a wide separation between the excitation and relaxation time scales (slow driving). We show in this paper, as a consequence of our more general analysis, that the dynamics of Zhang provide this separation naturally, and that an *infinitely slow driving* limit is actually reached as the size of the system goes to infinity. We then show, using a finite size scaling ansatz for the Lyapunov spectrum, that one can relate the obtained scaling exponent to the scaling of quantities such as avalanche size, duration, density of active sites, etc.

## II. DYNAMICAL SYSTEM DEFINITION AND BASIC PROPERTIES

### A. Definition

Zhang's model [5], widely inspired from the Bak-Tang-Wiesenfield precursor model [1], was introduced as a possible example of a model which ''self-organizes'' into a critical state in the thermodynamic limit, namely without fine tuning of a control parameter. Its beauty lies in its simplicity.

Let $\Lambda$ be a $d$-dimensional sublattice in $\mathbb{Z}^d$, taken as a square of edge length $L$ for simplicity. Call $N=|\mathcal{N}|=L^d$, where $|\mathcal{N}|$ denote the cardinality of a set, and let $^{\partial}\Lambda$ be the boundary of $\Lambda$, namely, the set of points in $\mathbb{Z}^d\setminus\Lambda$ at distance 1 from $\Lambda$. Each site $i\in\Lambda$ is characterized by its ''energy'' $X_i$, which is a non-negative real number. Call $\mathbf{X}=\{X_i\}_{i\in\Lambda}$ a configuration of energies. Let $E_c$ be a real, strictly positive number, called the *critical energy*, and $\mathcal{M}=[0,E_c[^N$. A configuration $\mathbf{X}$ is called ''stable'' iff $\mathbf{X}\in\mathcal{M}$ and ''unstable'' otherwise. If $\mathbf{X}$ is stable then one chooses a site $i$ at random with some probability $\nu_L(i)$, and adds to it an energy $\delta$, where $\delta$ is set to 1 in this paper (excitation). If a site $i$ is *overcritical* or *active* ($X_i\geqslant E_c$), it loses a part of its energy in equal parts to its $2d$ neighbors (relaxation). That is, we fix a parameter $\epsilon\in[0,1[$ such that, after relaxation of the site $i$, the remaining energy of $i$ is $\epsilon X_i$, while the $2d$ neighbors receive the energy $[(1-\epsilon)X_i]/2d$. Note, therefore, that there is a *local conservation of energy*. If several nodes are simultaneously active, the local distribution rules are additively superposed, i.e., the time evolution of the system is synchronous. The succession of updatings leading an unstable configuration to a stable one is called an *avalanche* (a more precise definition of an avalanche will be given below). There is dissipation at the boundaries: the sites of $^{\partial}\Lambda$ have always zero energy. As a result, all avalanches are *finite*. The addition of energy is *adiabatic*. When an avalanche occurs, one waits until it stops before adding a new energy quantum. Further excitations eventually generate a new avalanche, but, because of the adiabatic rule, each new avalanche starts from *only one* active site. Note that relaxation depends on *local* conditions while excitation is conditioned by *global* constraints (all sites are quiescent). It is conjectured that a critical state is reached, independently of $E_c$, at least for large $E_c$ values.[1]

### B. Zhang's model as a dynamical system

Because all avalanches are *finite* (for finite $L$), and since we are not interested in the transients, one can, without loss of generality take all initial energy configurations $\mathbf{X}\in\mathcal{M}$. All trajectories starting from $\mathcal{M}$ belong to a compact set $\mathcal{B}$. Call $\overline{\mathcal{M}}=\mathcal{B}\setminus\mathcal{M}$. $\overline{\mathcal{M}}$ contains a set of all unstable energy configurations achievable in an avalanche, starting from an energy configuration in $\mathcal{M}$.

Fix $\epsilon>0$, and call $\alpha=(1-\epsilon)/2d$. Let $h$ be the Heaviside function. Define $H$: $\mathbb{R}^N\to\{0,1\}^N$ such that $H(\mathbf{X})$ is the vector $\{h(X_i)\}_{i=1\ldots N}$. Call $\mathbf{X}_c$ the vector $\{E_c\}_{i=1\ldots N}$. Finally, let $\Delta$ be the discrete Laplacian. The dynamics is defined by the mapping $\mathbf{F}:\mathcal{B}\to\mathcal{B}$ such that

$$\mathbf{F}(\mathbf{X})=\mathbf{X}+\alpha\Delta[H(\mathbf{X}-\mathbf{X}_c)\cdot\mathbf{X}], \tag{1}$$

which redistributes the energy of the active sites in equal parts to the neighbors after one relaxation step. Note that $\mathbf{F}$ is the identity if no site is active, i.e. if $\mathbf{X}\in\mathcal{M}$, and that it is *piecewise linear* (i.e., linear on subdomains $\mathcal{B}_k\in\mathcal{B}$). $\mathbf{F}$ is a (singular) diffusion operator and $\alpha$ a diffusion coefficient.

[1]Strong deviations from a power law have been observed for small $E_c$ in one dimension [9].

It is useful to encode the dynamics of excitation in the following way. Let $\Sigma_\Lambda^+$ be the set of right infinite sequences $\mathbf{a}=\{a_1,\ldots,a_k,\ldots\}, a_k\in\Lambda$, and $\sigma$ be the *left shift* over $\Sigma_\Lambda^+$, namely, $\sigma\mathbf{a}=a_2a_3\ldots$. The elements of $\Sigma_\Lambda^+$ are called *excitation sequences*. The set $\Omega=\Sigma_\Lambda^+\times\mathcal{B}$ is the *phase space* of the Zhang's model, and $\hat{\mathbf{X}}=(\mathbf{a},\mathbf{X})$ is a point in $\Omega$. Zhang's model dynamics are given by a map of skew-product type $\hat{\mathbf{F}}:\Omega\to\Omega$, such that

$$\mathbf{X}\in\mathcal{M}\Rightarrow\hat{\mathbf{F}}(\hat{\mathbf{X}})=(\sigma\cdot\mathbf{a},\mathbf{X}+\mathbf{e}_a), \tag{2}$$

$$\mathbf{X}\in\bar{\mathcal{M}}\Rightarrow\hat{\mathbf{F}}(\hat{\mathbf{X}})=(\mathbf{a},\mathbf{F}(\mathbf{X})). \tag{3}$$

A knowledge of an initial energy configuration $\mathbf{X}$ and of a (infinite) sequence of excited sites $\mathbf{a}$ (of an initial condition $\hat{\mathbf{X}}$) fully determines the evolution. One can give $\Sigma_\Lambda^+$ a probability distribution $\nu_L$ corresponding to a random choice of excited sites. In Zhang's original model, the excited sites were chosen at random and independently with uniform probability. This corresponds to giving $\Sigma_\Lambda^+$ a *uniform Bernoulli measure*. Throughout this paper we will often think of the left Bernoulli shift on $\Sigma_\Lambda^+$ as represented by the system $z\to Nz \bmod 1$, $z\in[0,1]$.

In the following we will denote the two projections on the first and second coordinates by $\pi^u(\hat{\mathbf{X}})=\mathbf{a}$ and $\pi^s(\hat{\mathbf{X}})=\mathbf{X}$. The superscripts $u$ and $s$ mean *unstable* and *stable*, respectively, and correspond to the expansion (contraction) properties of the dynamics. Let $D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}$ be the tangent map of $\hat{\mathbf{F}}$ at $\hat{\mathbf{X}}$ and $D\hat{\mathbf{F}}^t_{\hat{\mathbf{X}}}$ the $t$th iterate. As shown below, $\pi^u(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})$ is expansive whereas $\pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})$ induces contraction. In the following we will use the notation $\hat{\mathbf{X}}(t)=\hat{\mathbf{F}}^t(\hat{\mathbf{X}})$ $[\mathbf{X}(t)=\pi^s(\hat{\mathbf{F}}^t(\hat{\mathbf{X}}))]$. Furthermore, note that $\pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})=D\mathbf{F}_{\mathbf{X}}$, and that $D\mathbf{F}_{\mathbf{X}}=I$, the identity matrix over $\mathbb{R}^N$, if $\mathbf{X}\in\mathcal{M}$.

Consider a point $\hat{\mathbf{X}}\in\Omega$. Its trajectory is intermittent, composed of bursts of excitation of the sites $a_1,a_2,\ldots a_n$, for those times $t$ such that $\mathbf{X}(t)\in\mathcal{M}$, followed by relaxation periods when $\mathbf{X}(t)\in\bar{\mathcal{M}}$. Define the following hierarchy of *waiting times*:

$$\gamma_0(\hat{\mathbf{X}})=0, \tag{4}$$

$$\sigma_i(\hat{\mathbf{X}})=\inf_{t>\gamma_{i-1}}\{\mathbf{X}(t)\in\bar{\mathcal{M}}\},\quad i\geqslant 1 \tag{5}$$

$$\gamma_i(\hat{\mathbf{X}})=\inf_{t>\sigma_i}\{\mathbf{X}(t)\in\mathcal{M}\},\quad i\geqslant 1 \tag{6}$$

For $i\geqslant 1$, $\sigma_i(\hat{\mathbf{X}})$ $[\gamma_i(\hat{\mathbf{X}})]$ is the starting time (ending time) of the $i$th avalanche occurring during the evolution of $\hat{\mathbf{X}}$. Therefore, the *avalanche duration* of the $i$th avalanche is

$$\tau_i(\hat{\mathbf{X}})=\gamma_i(\hat{\mathbf{X}})-\sigma_i(\hat{\mathbf{X}}). \tag{7}$$

In the same way, one defines

$$\omega_i(\hat{\mathbf{X}})=\sigma_i(\hat{\mathbf{X}})-\gamma_{i-1}(\hat{\mathbf{X}}), \tag{8}$$

which is the number of excitations between the end of the avalanche $i-1$ and the beginning of the next avalanche. In this way, one naturally introduces two time scales: a *local time t* corresponding to one step of iteration in the dynamics, and an *avalanche time* $\tau_i$ corresponding to the duration of an avalanche (a similar description was used in Ref. [13]).

The waiting times are useful for defining the usual avalanche observables. The number $|\mathcal{N}|$ of relaxing sites for a given configuration is

$$r(\hat{\mathbf{X}})=|\mathcal{N}|\{i\in\Lambda,\ X_i\geqslant E_c\}. \tag{9}$$

The *avalanche size* is

$$s(\hat{\mathbf{X}})=\sum_{t=1}^{\tau(\hat{\mathbf{X}})} r(\hat{\mathbf{F}}^t(\hat{\mathbf{X}})), \tag{10}$$

where

$$\tau(\hat{\mathbf{X}})=\inf_{t\geqslant 1}\{\mathbf{F}^t(\mathbf{X})\in\mathcal{M}\}-1 \tag{11}$$

is the duration of the avalanche that occurs when exciting the site $a_1$ in a stable energy configuration $\mathbf{X}$. It is zero if one drops energy without relaxation.

The structure of an avalanche can be encoded by the sequence of active sites $A(\hat{\mathbf{X}})=\{A_t(\hat{\mathbf{X}})\}_{1\leqslant t\leqslant\tau(\hat{\mathbf{X}})}$ where $A_t(\hat{\mathbf{X}})=\{j\in\Lambda|X_j(t)\geqslant E_c\}$. [Note that $A_1(\hat{\mathbf{X}})$ is nonempty and equal to $\{a_1\}$ iff $\mathbf{X}+\mathbf{e}_{a_1}$ is active.] Correspondingly, there exists a partition[2] of $\Sigma_\Lambda^+\times\mathcal{M}$ into domains $\mathcal{P}_{i,k}=[i]\times\mathcal{M}_{i,k}$, where $[i]$ is the set of sequences in $\Sigma_\Lambda^+$ having a first digit $i$, such that for any energy configuration $\mathbf{X}\in\mathcal{M}_{i,k}$ the excitation of site $i$ leads to the same avalanche (the same sites relax at the same time). Under some moderate assumptions (see Ref. [11]), this allows us to define a symbolic coding for the avalanche and a transition graph, giving the transition rules between successives avalanches, and to show that the dynamical system admits a unique, fractal, invariant set. The boundary of the domains $\mathcal{P}_{i,k}$ constitutes the *singularity set* of $\hat{\mathbf{F}}$, called $\mathcal{S}$. This is the set of points where $\hat{\mathbf{F}}$ is not continuous.

### C. Stationary state and probability of avalanche observables

Let $\hat{\mu}_L$ be an invariant measure for the dynamical system $\{\Omega,\hat{\mathbf{F}}\}$, where $L$ refers to the lattice size, namely, $\hat{\mu}_L(\hat{\mathbf{F}}^{-1}(\mathcal{A}))=\hat{\mu}_L(\mathcal{A})$, where $\mathcal{A}\in\Omega$ is a measurable set. Since $\Omega$ has a product structure where the unstable foliation is always transverse to the stable one, and since the dynamical system is a skew product, $\hat{\mu}_L=\nu_L\times\mu_L$, where $\nu_L$ is the induced measure on the unstable direction or *excitation* measure, and $\mu_L$ is the induced measure on $\mathcal{B}$ or measure on the *energy* configurations. For simplicity we will assume that $\nu_L$

[2]This partition is induced by the partition of $\mathcal{B}$ into domains of continuity for $\mathbf{F}$ [11].

is a Bernoulli measure, namely, that the successive excited sites are chosen *independently* with fixed rates. Once we have fixed the distribution of excitation, we are interested in the possible $\mu_L$ measures. Of special physical importance are the measures obtained by iterating the Lebesgue measure $\mu_{Leb}$ (see footnote 3) on $\mathcal{M}$, that is, $\lim_{n\to\infty}(1/n)\Sigma_{i=0}^{n-1}\hat{\mathbf{F}}^i(\nu_L\times\mu_{Leb})$. When the excitation measure $\nu_L$ is itself the Lebesgue measure on $[0,1]$ (corresponding to choosing the excited sites with uniform probability) the measure obtained is called the Sinai-Ruelle-Bowen (SRB) measure. More generally, we will call the (conditional) SRB measure $\lim_{n\to\infty}(1/n)\Sigma_{i=0}^{n-1}\hat{\mathbf{F}}^i(\nu_L\times\mu_{Leb})$, for a fixed $\nu_L$. This is a natural invariant measure from the physical point of view, since it gives the ensemble average with respect to typical initial energy configurations.

It is common in SOC literature to assume ergodicity. In our setting, the physically relevant ergodic property is equivalent to assuming that the SRB measure is unique. Proving the ergodicity in Zhang's model is clearly a difficult task, which is beyond the scope of this paper. However, we note that this point was discussed in a previous paper [11], where strong mathematical arguments in favor of this were given. Actually, ergodicity was proved, but restricted to the one-dimensional model and to some $E_c$ interval. A general proof is under construction, and will be published elsewhere [14]. On physical grounds, note that the failure of ergodicity would lead to a stationary state depending on initial conditions. This would contradict the implicit SOC assumption that the stationary state is unique. In the following, we will therefore assume that ergodicity holds and that $\hat{\mu}_L$ is the unique SRB measure. This implies, in particular, an almost-sure equality between the ensemble average and the time average: if $\phi$ is some observable (a function $\Omega\to\mathbb{R}$, integrable with respect to $\hat{\mu}_L$),

$$\overline{\phi}_L \overset{\text{def}}{=} \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\phi(\hat{\mathbf{F}}^t(\hat{\mathbf{X}}))=\int_\Omega \phi(\hat{\mathbf{X}})d\hat{\mu}_L(\hat{\mathbf{X}}) \overset{\text{def}}{=} E_L[\phi] \tag{12}$$

for a typical (namely Lebesgue almost surely) initial condition $\hat{\mathbf{X}}$. Here and in the following, $\overline{\ }L$ will denote the time average, while $E_L[\ ]$ will be an ensemble average on a lattice of size $L$.

From a dynamical system point of view, $\hat{\mu}_L$ is the natural object to deal with. However, in the SOC literature one is more interested in the probability distribution of some avalanche observable and its scaling properties in the thermodynamic limit. Fix an avalanche observable, say $s$. Call $\mathcal{P}_s$ the union of domains $\mathcal{P}_{i,k}$ such that avalanches corresponding to each domain $\mathcal{P}_{i,k}$ have the same size $s$. Then the probability of having an avalanche of size $s$, by excitation of a *stable* configuration, is $\text{Prob}[s(\hat{\mathbf{X}})=s|\hat{\mathbf{X}}\in\mathcal{P}]=[\hat{\mu}_L(\mathcal{P}_s)/\hat{\mu}_L(\mathcal{P})]$ $=[\hat{\mu}_L(\mathcal{P}_s)/\mu_L(\mathcal{M})]$. In this definition we include the avalanches of size zero (excitation without relaxation). However, it is more natural from the SOC point of view to exclude this case. We therefore define $P_L(s)$ as the probability of having an avalanche of size $s$ *strictly larger than 0*.[4]

$$P_L[s]\overset{\text{def}}{=}\frac{\hat{\mu}_L(\mathcal{P}_s)}{p_L},\quad s\geqslant 1, \tag{13}$$

where $p_L\overset{\text{def}}{=}\text{Prob}[s(\hat{\mathbf{X}})\geqslant 1,\mathbf{X}\in\mathcal{M}]$ is the probability of *initiating* an avalanche. The average with respect to $P_L[s]$, denoted further on by $\langle\ \rangle_L$, is

$$\langle\psi(s)\rangle_L\overset{\text{def}}{=}\sum_{s=1}^{\xi_L^s}P_L[s]\psi(s), \tag{14}$$

where $\psi$ is some real function, and $\xi_L^s$ is the maximal value that the observable $s$ can have on a lattice of size $L$ (note that $\xi_L^s$ also depends on $E_c$, $\epsilon$, and $d$, but is *bounded* if $L<\infty$). The same definition holds for any other avalanche observable. From the ergodic theorem

$$\langle\psi(s)\rangle_L=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\psi(s_i), \tag{15}$$

where $s_i$ is the size of the $i$th avalanche occuring in the trajectory of a generic point $\hat{\mathbf{X}}$.

One has

$$p_L=\hat{\mu}_L[\cup_{i=1}^{N}\{a_1=i,X_i\in[E_c-1,E_c[\}]=\sum_{i=1}^{N}p_L(i), \tag{16}$$

where

$$p_L(i)\overset{\text{def}}{=}\nu_L(i)\mu_L\{X_i\in[E_c-1,E_c[\} \tag{17}$$

is the probability that *an avalanche starts at a site* $i$. Note that the probabilities $p_L(i)$ *depend a priori* on $i$ even if the excitation measure is uniform. In this case, however, Eq. (16) reduces to

$$p_L=\frac{1}{N}\sum_{i=1}^{N}\mu_L\{X_i\in[E_c-1,E_c[\}. \tag{18}$$

Fix $\hat{\mathbf{X}}$ and $T$, call $n(T,\hat{\mathbf{X}})$ the number of *complete* avalanches occurring until local time $T$ for the initial condition $\hat{\mathbf{X}}$. Obviously, $n(T,\hat{\mathbf{X}})\to\infty$ as $T\to\infty$, $\forall\mathbf{X}$. Then, from the ergodic theorem,

[3]Or any absolutely continuous measure, which corresponds to selecting the initial energy configuration with a probability distribution having a density.

[4]In view of the expected critical behavior as $L\to\infty$, one usually writes a scaling form $P_L(s)=f_L(s)/s^{\tau_s}$ where $f_L(s)$ is a cutoff term accounting for finite size effects on large scales. $P_L(s)$ is not defined for $s=0$ unless we assume very special properties for $f_L(s)$.

$$p_L = \lim_{T\to\infty} \frac{n(T,\hat{\mathbf{X}})}{T}. \tag{19}$$

One can decompose $T$ as $T=\Sigma_{i=1}^{n(T,\hat{\mathbf{X}})}\tau_i+\Sigma_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i+K(\hat{\mathbf{X}})$, where $K(\hat{\mathbf{X}})$ is some residual time, finite, whatever $T$ and $\hat{\mathbf{X}}$ [$K(\hat{\mathbf{X}})$ is bounded by the largest avalanche duration]. Note that $\tau_i(\omega_i)$ stands for $\tau_i(\hat{\mathbf{X}})$ [$\omega_i(\hat{\mathbf{X}})$], but we removed the $\hat{\mathbf{X}}$ dependence in order to simplify the notations. Then, as $T$ goes to infinity,

$$\frac{n(T,\hat{\mathbf{X}})}{T} \sim \frac{n(T,\hat{\mathbf{X}})}{\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\tau_i+\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i}$$

$$= \frac{n(T,\hat{\mathbf{X}})}{\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\tau_i} - \frac{n(T,\hat{\mathbf{X}})}{\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i}\frac{\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i}{T}.$$

Call

$$\overline{\omega_L} \overset{\text{def}}{=} \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i = \mu_L(\mathcal{M}) \tag{20}$$

the *probability of dropping energy in the system at a given time* (the equality holds for $\mu_L$ for almost every $\hat{\mathbf{X}}$ from the ergodic theorem). $\overline{\omega_L}(i)=\mathrm{Prob}[a_1=i,\mathbf{X}\in\mathcal{M}]=\nu_L(i)\overline{\omega_L}$ is the probability of dropping energy on site $i$, at a given time, and is called the *driving rate* in the literature [15]. One has

$$p_L = \frac{1-\overline{\omega_L}}{\langle\tau\rangle_L} = \frac{\mu_L(\overline{\mathcal{M}})}{\langle\tau\rangle_L}, \tag{21}$$

where $\langle\tau\rangle_L$ is the *average avalanche duration*.

There exists an important relation linking the avalanche averages (average with respect to $P_L$) to the local time average (average with respect to $\hat{\mu}_L$). Let $\phi:\Omega\to\mathbb{R}$ be some observable *such that* $\phi(\hat{\mathbf{X}})=0$ *whenever* $\mathbf{X}\in\mathcal{M}$. A related avalanche observable can be defined by summing the values that $\phi$ takes in one avalanche. That is, call $f_i(\hat{\mathbf{X}})=\Sigma_{t=\sigma_i(\hat{\mathbf{X}})}^{\gamma_i(\hat{\mathbf{X}})}\phi(\hat{\mathbf{X}}(t))$. [An important example is when $\phi(\hat{\mathbf{X}})=r(\hat{\mathbf{X}})$, the number of active sites in one step. Then $f_i(\hat{\mathbf{X}})$ is the size of the $i$th avalanche in the trajectory of $\hat{\mathbf{X}}$.] One obtains

$$\overline{\phi_L} = \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{n(T,\hat{\mathbf{X}})}\sum_{t=\sigma_i(\hat{\mathbf{X}})}^{\gamma_i(\hat{\mathbf{X}})}\phi(\hat{\mathbf{X}}(t)),$$

which yields

$$\overline{\phi_L} = p_L\langle f\rangle_L. \tag{22}$$

In particular,

$$\overline{r_L} = p_L\langle s\rangle_L \tag{23}$$

Finally we define the probability that a site $i$ is active (often called the *density of active sites* in the literature[5]):

$$\rho_L(i) \overset{\text{def}}{=} \mu_L[X_i \geqslant E_c] \tag{24}$$

and

$$\rho_L^{av} = \frac{1}{N}\sum_{i=1}^{N}\rho_L(i). \tag{25}$$

$\rho_L^{av}$ is believed to act as an order parameter in Zhang's model.

## III. DYNAMICAL PROPERTIES AND LYAPUNOV EXPONENTS

### A. Jacobian matrix and Lyapunov exponents

Due to the piecewise affine structure of the map $\mathbf{F}$, the Jacobian matrix $DF_{\mathbf{X}}$ plays a central role in Zhang's model, since it characterizes the energy transport. Indeed, the entry $DF^t_{\mathbf{X},ij}$ is the ratio of energy flowing from site $j$ to site $i$ in $t$ times steps for the initial condition $\mathbf{X}$. Define $Z_k(\mathbf{X}(t))=H(X_k(t)-E_c)$. This is a random variable, taking a value 0 if $X_k(t)$ is stable, and a value 1 otherwise, whose probability distribution is induced (at stationarity) by the invariant measure $\hat{\mu}_L$. More precisely, $\mathrm{Prob}[Z_k(\mathbf{X}(t))=1]=\rho_L(k)$. Let $\mathbf{Z}(\mathbf{X})=\{Z_k(\mathbf{X})\}_{k=1}^N$, and call $S(\mathbf{X})=\Delta\mathbf{Z}(\mathbf{X})I$ [equivalently $S(\mathbf{X})$ is the matrix of entries $S_{ij}(\mathbf{X})=\Delta_{ij}Z_j(\mathbf{X})$]. $S$ is the ''toppling'' operator of Zhang's model. The Jacobian matrix is $DF_{\mathbf{X}}=I+\alpha S(\mathbf{X})$, while $DF^t_{\mathbf{X}}$ is given by

$$DF^t_{\mathbf{X}} = I+\alpha\sum_{t_0=1}^{t}S(\mathbf{X}(t_0))+\alpha^2\sum_{t\geqslant t_1>t_0\geqslant 1}S(\mathbf{X}(t_1))S(\mathbf{X}(t_0))$$

$$+\cdots+\alpha^r\sum_{t\geqslant t_{r-1}>t_{r-2}\cdots>t_0\geqslant 1}S(\mathbf{X}(t_{r-1}))$$

$$\times S(\mathbf{X}(t_{r-2}))\cdots S(\mathbf{X}(t_0))+\cdots+\alpha^t S(\mathbf{X}(t))$$

$$\times S(\mathbf{X}(t-1))\cdots S(\mathbf{X}(1)). \tag{26}$$

Therefore, the generic term (say of order $r$) is a ''propagator'' transmitting the energy in $r$ times steps. Note that this formula is exact. It calls for the following remarks.

(i) The maps $S(\mathbf{X})$ do not commute, and they depend on the state. This is a key difference from Dhar's model, since it induces a *non-Abelian* structure and a ''toppling'' operator depending *not only on the site, but also on the whole energy configuration*. In particular, the propagator *is not a mere polynomial of the Laplacian*.

[5]We will keep this terminology throughout this paper, though $\rho_L(i)$ is not, strictly speaking, a density since $\Sigma_{i=1}^N\rho_L(i)\neq 1$.

(ii) The evolution depends *a priori* on the whole trajectory, and therefore the strong memory effects expected in a critical phenomenon can be treated from Eq. (26).

If one defines the excitation times for a given trajectory by

$$\nu_k(\hat{\mathbf{X}}) = \inf_{t>\nu_{k-1}(\hat{\mathbf{X}})} \{\mathbf{X}(t) \in \mathcal{M}\}, \tag{27}$$

with $\nu_0=\gamma_0=1$, the energy configuration at time $T$, for an initial condition $\hat{\mathbf{X}}$ is

$$\mathbf{X}(T) = DF^T_{\mathbf{X}} \cdot \mathbf{X} + \sum_{i=1}^{m(T,\hat{\mathbf{X}})} DF^{t-\nu_i(\hat{\mathbf{X}})}_{\mathbf{X}} \cdot \mathbf{e}_{a_{\nu_i(\hat{\mathbf{X}})}}, \tag{28}$$

where $m(T,\hat{\mathbf{X}})$ is the number of excitations on a time interval of length $T$ for the initial condition $\hat{\mathbf{X}}$. The first term corresponds to a redistribution of the initial energy configuration, and the second one to a redistribution of the energy quantum $\delta=1$ dropped in the system at times $\nu_k(\hat{\mathbf{X}})$. Since the equilibrium average is assumed to be independent of the initial condition, the first term has to decay to zero as $t\to\infty$, with a decay rate corresponding to the characteristic relaxation time to equilibrium.

It is therefore important to understand well the (spectral) properties of the $DF^t\mathbf{X}$ in the infinite time limit. Were $S(\mathbf{X})$ to be the Laplace operator, then were the spectrum of $DF^t\mathbf{X}$ to be composed of Fourier modes, and the relaxation time to equilibrium would be the slowest mode. However, the mere presence of a singular term $Z(\mathbf{X})$ certainly makes a big difference. Since $S$ depends on $\mathbf{X}$ one clearly has to study the decay rates averaged on a full (typical) trajectory, or equivalently one has to compute the ensemble average. In this view, the law of the stochastic process $\{Z(\mathbf{X}(t))\}_{t=0}^{+\infty}$ (namely, the density of active sites and correlations at all times) certainly plays a role.

The numbers characterizing the decay (expansion) rates of the norm of a small pertubation in the trajectory's tangent space of a point $\hat{\mathbf{X}}$ under the action of the infinite product matrix $DF^t\hat{\mathbf{X}}$, $t\to\infty$, are the *Lyapunov exponents*. They are *real* numbers, well defined under some moderate assumptions on $DF\hat{\mathbf{X}}$ (see Ref. [16]), and are almost surely independent of $\hat{\mathbf{X}}$. Furthermore they are also independent of the norm (in finite dimension).

As shown in Ref. [11] and widely discussed in this paper, all the Lyapunov exponents are different from zero, for *finite* $L$ (weak hyperbolicity). One remarkable consequence is that the asymptotic dynamics lie on a fractal attractor, and that the Lyapunov spectrum is closely related to the (local) fractal properties of the invariant set through the Kaplan-Yorke [17] and Ledrappier-Young formulas [18,11]. At this point a remark is necessary. Hyperbolicity is clearly a particular feature of Zhang's model, and of similar models where the amount of redistributed energy from a critical site $i$ depends on its energy $X_i$. Conversely, in models like that of BTW or Dhar, the amount of transferred energy is a constant. As a direct consequence, in these models, the dynamics is simply a piecewise translation in phase space, and the uniform measure is preserved [4]. Hence all Lyapunov exponents are zero [19]. There is therefore clearly a *structural difference* between the dynamics of Zhang-type models and sandpiles. At first sight this observation seems to ruin the hope of classifying Zhang-type models and sandpile models in the same ''universality class.'' However, in this paper we show that the hyperbolicity of Zhang's model is lost in the thermodynamic limit. That is, some of the Lyapunov exponents go to zero as $L\to\infty$, with a polynomial rate (exponent $\tau_\lambda$ in Sec. II) closely related to SOC critical exponents. It might therefore well be that these two classes of model share the same SOC critical exponents in the thermodynamic limit, though their dynamics are still of different natures, even in this limit.

Due to the skew product structure, the tangent map at any point $\hat{\mathbf{X}}$ admits a natural splitting $D\hat{\mathbf{F}}=(\pi^u(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}), \pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}))$, where the one-dimensional map $\pi^u(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}})$ is expansive. Indeed, the average expansion rate is given by

$$\lambda_L(0) = \lim_{T\to\infty} \frac{1}{T} \log[\det(\pi^u(D\hat{\mathbf{F}}^T_{\hat{\mathbf{X}}}))] = \overline{\omega}_L \log(N), \tag{29}$$

since $\det(\pi^u(D\hat{\mathbf{F}}^T_{\hat{\mathbf{X}}})) = N^{\Sigma_{i=1}^{n(T,\hat{\mathbf{X}})}\omega_i(\hat{\mathbf{X}})}$. Therefore, since $\overline{\omega}_L \neq 0$, there is a *positive Lyapunov exponent* in the dynamics. Note that this is due to the excitation rule, and that it simply reflects the ''chaotic'' properties of the Bernoulli shift.

A more important issue concerns $\pi^s(D\hat{\mathbf{F}}_{\hat{\mathbf{X}}}) = D\mathbf{F}_{\mathbf{X}}$. The Oseledec theorem [16] asserts that under mild conditions on $D\mathbf{F}_{\mathbf{X}}$ there exists a hierarchy of Lyapunov exponents $\lambda_L(1) > \cdots \lambda_L(N)$, which are almost surely constant with respect to the Lebesgue measure, and a hierarchy of nested subspaces (Oseledec splitting)

$$\mathbb{R}^N = \mathcal{V}_1(\hat{\mathbf{X}}) \supset \mathcal{V}_2(\hat{\mathbf{X}}) \supset \mathcal{V}_N(\hat{\mathbf{X}})$$

depending on $\hat{\mathbf{X}}$, such that the norm of a perturbation $\mathbf{v} \in \mathcal{V}_i(\hat{\mathbf{X}}) - \mathcal{V}_{i+1}(\hat{\mathbf{X}})$ is given by

$$\|D\mathbf{F}^t_{\mathbf{X}} \cdot \mathbf{v}\| = C(\hat{\mathbf{X}},t) e^{\lambda_L(i)\cdot t} \|\mathbf{v}\|, \tag{30}$$

where $\lim_{t\to\infty}(1/t)\log C(\hat{\mathbf{X}},t) = 0$ almost surely; that is, $\lambda_L(i)$ is the exponential rate of variation of $\|\mathbf{v}\|$. Define $M(\mathbf{X},t) = \tilde{D}\mathbf{F}^t_{\mathbf{X}} \cdot D\mathbf{F}^t_{\mathbf{X}}$ and $\Lambda = \lim_{t\to\infty} M(\mathbf{X},t)^{1/2t}$ (the Oseledec multiplicative ergodic theorem insures that this limit exists almost surely, and is a constant). Then the Lyapunov exponents are the logarithm of the eigenvalues of $\Lambda$. $M(\mathbf{X},t)$ being symmetric, it admits an orthogonal basis $\{\mathbf{v}_i(\mathbf{X},t)\}_{i=1}^N$, and eigenvalues $\mu_i(\mathbf{X},t)$ such that $\lambda_L(i) = \lim_{t\to\infty}(1/2t)\log(\mu_i(\mathbf{X},t))$. Furthermore, each $\mathbf{v}_i(\mathbf{X},t)$ converges exponentially to a vector $\mathbf{v}_i(\hat{\mathbf{X}})$ in $\mathbb{R}^N$, depending on $\hat{\mathbf{X}}$ [20]. $\mathbf{v}_i(\hat{\mathbf{X}})$'s therefore constitutes a basis for the Oseledec splitting. We call these *Oseledec modes* for the trajectory of $\hat{\mathbf{X}}$. They can be numerically obtained from the QR decomposition used in the computation of the Lyapunov spectrum

(see Ref. [17]). It was shown in Ref. [11] that $\lambda_L(i)$ are all negative for finite $L$, namely, all vectors in $\mathbb{R}^N$ are asymptotically contracted.

From this discussion, one expects a close connection between the Lyapunov spectrum and the energy transport in Zhang's model. In particular, the following formula can be proved [11]:

$$\sum_{i=1}^{N} \lambda_L(i) = \log(\epsilon)(1-\overline{\omega}_L)\frac{\langle s\rangle_L}{\langle \tau\rangle_L} = p_L \log(\epsilon)\langle s\rangle_L . \quad (31)$$

This relates the Lyapunov spectrum, which characterizes *local* properties of the *microscopic* dynamics, to the avalanche *statistical* properties of the *macroscopic* system. Note that the exponent $\lambda_L(i)$ gives the contraction rate in the direction $\mathbf{v}_i(\hat{\mathbf{X}})$ versus the *local time*. One can also define the average contraction *per avalanche*, $\chi_L(i)$, given from Eq. (22) by

$$\chi_L(i) = \frac{\langle \tau\rangle_L}{1-\overline{\omega}_L}\lambda_L(i) = \frac{\lambda_L(i)}{p_L}. \quad (32)$$

Then the sum of $\chi_L$'s, giving the average volume contraction *per avalanche*, is related to the average avalanche size by

$$\sum_{i=1}^{N} \chi_L(i) = \log(\epsilon)\langle s\rangle_L . \quad (33)$$

Note that $\langle s\rangle_L$ corresponds to the total energy transport within one avalanche, and is believed to be related to the total response function [15]. Our formula shows that it is also equal to the volume contraction in phase space produced on average by one avalanche.

### B. Oseledec modes

To each negative Lyapunov exponent $\lambda_L(i)$, $i=1,\ldots,N$, is associated a characteristic time $t_L(i)=|\lambda_L(i)|^{-1}$, the time for of a perturbation in the Oseledec direction $i$ to vanish. This therefore define a hierarchy

$$t_L(1) > t_L(2) > \cdots t_L(N).$$

Note that there is no contradiction with the expected critical behavior in the thermodynamic limit, since as $L\to\infty$ there are an infinite number of characteristic time scales.

From a physical point of view a perturbation can be viewed as a small modification of the initial energy landscape $\mathbf{X}$. It can be localized (for example one site perturbed) or spread. The attenuation is due to two distincts effects: propagation through the lattice, and dissipation at the boundaries.

Note that according to the Oseledec mode under consideration, the contraction can be due (on average) to the effect of one avalanche [if $t_L(i)$ is small compared to the average avalanche size], or to the cumulative effect of many avalanches [if $t_L(i)$ is large]. The coefficient $\chi_L(i)$ [Eq. (32)] gives the average contraction per avalanche for the $i$th Oseledec mode. Therefore the number $1/\chi_L(i)$ gives an estimate of the number of avalanches needed to have a reduction of the initial perturbation of a ratio $1/e$ for this mode. Therefore, a crossover point can be estimated by

$$\chi_L(i_c) \sim 1. \quad (34)$$

We will call Oseledec modes corresponding to $\lambda_L(i) \ll \lambda_L(i_c)$ $[\lambda_L(i) \gg \lambda_L(i_c)]$ slow (fast).

#### 1. Bounds on the first negative Lyapunov exponent

We give a bound on the first Lyapunov exponent related to the energy dissipation at the boundaries. Call $\Phi_j^{out}(t,\hat{\mathbf{X}}) \overset{\text{def}}{=} 1-\Sigma_{i=1}^{N} DF^t_{\mathbf{X},ij}$. Since the energy is locally conserved, $\Phi_j^{out}(t,\hat{\mathbf{X}})$ is the ratio of the initial energy $X_j$ given by the site $j$ to the boundary $^{\partial}\Lambda$ in $t$ time steps. In other words, the energy coming from $X_j$ and lost at time $t$ is $\Phi_j^{out}(t,\hat{\mathbf{X}})X_j$. The following holds.

*Proposition 1*: The largest negative Lyapunov exponent $\lambda_L(1)$ admits the following bounds:

$$0 > \lim_{t\to\infty} \frac{1}{t}\log[1-\min_j(\Phi_j^{out}(t,\hat{\mathbf{X}}))]$$

$$\geqslant \lambda_L(1) \geqslant \lim_{t\to\infty}\frac{1}{t}\log[1-\max_j \Phi_j^{out}(t,\hat{\mathbf{X}})]. \quad (35)$$

This is interpreted as follows. As $t\to\infty$, $\Phi_j^{out}(t,\hat{\mathbf{X}})\to 1$, $\forall j$, $\forall \hat{\mathbf{X}}$, since, eventually, all the initial energy coming from $\mathbf{X}$ is lost at the boundaries. The limit $\lim_{t\to\infty}(1/t)\log[1-\Phi_j^{out}(t,\hat{\mathbf{X}})]$ gives the exponential rate of convergence of $\Phi_j^{out}(t,\hat{\mathbf{X}})$ to 1. In other words, it gives the exponential decrease for the ratio of the initial energy still in the lattice at a given time. The maximal negative Lyapunov exponent is bounded by the extremal dissipation rates. One sees, therefore, that the contraction in the principal Oseledec mode is mainly due to the dissipation at the boundaries. We shall see later that $\lambda_L(1)$ is essentially related to the so-called *dissipation rate*.

*Proof*: It is easy to show that there exists a time $t_s$ depending on $E_c$, $\epsilon$, and $d$ such that, whatever the value of $\hat{\mathbf{X}}$ each site in the lattice has relaxed at least once after this time, and therefore all sites of the boundary have dissipated energy. At time $t$ the energy coming from site $j$ with initial energy $X_j$, and redistributed into the lattice, is $\Sigma_{i=1}^{N} DF^t_{\mathbf{X},ij}X_j$. For $t\geqslant t_s$, $\Phi_j^{out}(t,\hat{\mathbf{X}})>0$, and $\Sigma_{i=1}^{N} DF^t_{\mathbf{X},ij}$ is bounded away from 1. Since $D\mathbf{F}^t_{\hat{\mathbf{X}}}$ is a matrix with positive entries,

$$\min_j\left(\sum_{i=1}^{N} D\mathbf{F}^t_{\mathbf{X},ij}\right) = 1-\max_j \Phi_j^{out}(t,\hat{\mathbf{X}}) \leqslant \rho(D\mathbf{F}^t_{\mathbf{X}})$$

$$\leqslant \max_j\left(\sum_{i=1}^{N} DF^t_{\mathbf{X},ij}\right)$$

$$= 1-\min_j \Phi_j^{out}(t,\hat{\mathbf{X}})$$

$$< 1, \quad (36)$$

where $\rho(D\mathbf{F}^t_{\mathbf{X}})$ is the spectral radius of $D\mathbf{F}^t_{\mathbf{X}}$.

The largest negative Lyapunov exponent is given by

$$\lambda_L(1)=\lim_{t\to\infty}\frac{1}{t}\log(\|DF^t_{\mathbf{X}}\|_2), \tag{37}$$

where $\|\ \|_2$ is the $L_2$ norm. In Eq. (37) the limit does not depend on $\hat{\mathbf{X}}$, provided $\hat{\mathbf{X}}$ belongs to the support of $\hat{\mu}_L$. One has $\rho(D\mathbf{F}^t_{\mathbf{X}})\leqslant\|DF^t_{\mathbf{X}}\|_2$, and, therefore,

$$\lambda_L(1)\geqslant\lim_{t\to\infty}\frac{1}{t}\log[1-\max_j\Phi^{out}_j(t,\hat{\mathbf{X}})].$$

Furthermore, all norms being equivalent in finite dimensions, Eq. (37) also holds for the $L_1$ norm where

$$\|DF^t_{\mathbf{X}}\|_1=\sup_{\mathbf{X}}\frac{\sum_{i,j=1}^{N}D\mathbf{F}^t_{\mathbf{X},ij}|X_j|}{\sum_{j=1}^{N}|X_j|}.$$

$D\mathbf{F}^t_{\mathbf{X},ij}$'s being positive, the supremum is certainly achieved for positive $X_i$ values. Therefore,

$$\|DF^t_{\mathbf{X}}\|_1=\sup_{\mathbf{X}}\frac{\sum_{j=1}^{N}[1-\Phi^{out}_j(t,\hat{\mathbf{X}})]X_j}{\sum_{j=1}^{N}X_j}$$

$$=1-\inf_{\mathbf{X}}\frac{1}{\sum_{j=1}^{N}X_j}\sum_{j=1}^{N}\Phi^{out}_j(t,\hat{\mathbf{X}})X_j$$

$$\leqslant 1-\inf_{\mathbf{X}}\min_j(\Phi^{out}_j(t,\hat{\mathbf{X}}))$$

The limit $\lim_{T\to\infty}\log[1-\min_j(\Phi^{out}_j(t,\hat{\mathbf{X}}))]$ is a constant for any $\hat{\mathbf{X}}$ in support of $\hat{\mu}_L$. Hence

$$\lambda_L(1)\leqslant\lim_{t\to\infty}\frac{1}{t}\log[1-\min_j(\Phi^{out}_j(t,\hat{\mathbf{X}}))]$$

#### *2. Stabilizing modes*

The contraction in the principal Oseledec mode (the first negative Lyapunov exponent) is mainly due to the dissipation at the boundaries. On the other hand, it is possible to have a large contraction in one local time step without reaching the boundaries. Indeed, the tangent matrix $DF_{\mathbf{X}}$ has the following property, which can be checked by direct computation.

*Proposition 2*: Let $\Lambda=\Lambda_c(\mathbf{X})\oplus\Lambda_n(\mathbf{X})$, where $\Lambda_c(\mathbf{X})=\{i\in\Lambda|X_i\geqslant X_c\}$, $\Lambda_n(\mathbf{X})=\{i\in\Lambda|X_i<X_c\}$, and $n_c(\mathbf{X})=\mathcal{N}\Lambda_c(\mathbf{X})$; then $DF_{\mathbf{X}}$ has $n_c(\mathbf{X})$ eigenvalues $\epsilon$ corresponding to the eigenvectors

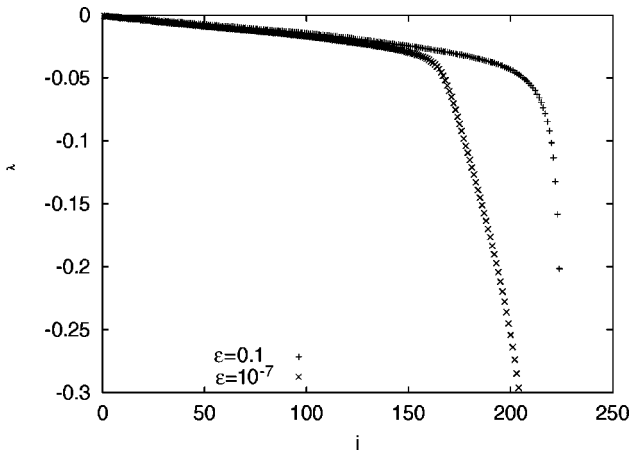


FIG. 1. Lyapunov spectrum for $\epsilon=0.1$ and $10^{-7}$, $L=15$, and $E_c=2.2$.

$$k_i(\mathbf{X})=2d\mathbf{e}_i-\sum_{j\in\mathcal{U}_i}\mathbf{e}_j,\quad i\in\Lambda_c(\mathbf{X}), \tag{38}$$

where $\mathcal{U}_i$ denotes the set of sites in $\Lambda$ at distance 1 from $i$. There are also $N-n_c(\mathbf{X})$ neutral eigenvalues associated with the eigenvectors $\mathbf{e}_i$, $i\in\Lambda_n(X)$.

The eigenvectors $k_i$ produce arbitrarily large contraction as $\epsilon\to 0$. In particular, in the original Zhang model, where $\epsilon=0$, they correspond to *kernel* modes, which have eigenvalues 0. Note that, in this case, the dimension of the kernel of the product tangent map $DF^t_{\mathbf{X}}$ increases with $t$. However, it is strictly lower than $N$ as $t\to\infty$ [11]. It is easy to check that these modes have zero energy, except if some of the $\mathbf{e}_i$'s correspond to sites neighboring the boundary. This occurs with small (but nonzero) probability. These modes act as directions where a single local time step is sufficient to reduce the initial perturbation by a factor $\epsilon$, with a small variation of the total energy on average. They correspond dynamically to directions transverse to the attractor, and their contraction corresponds to a fast convergence onto the attractor. For this reason we call them *stabilizing modes*. In the Lyapunov spectrum they can be identified because the corresponding Lyapunov exponents go to $-\infty$ as $\epsilon\to 0$, while the other part of the spectrum weakly depends on $\epsilon$ (see Fig. 1).

#### *3. Transport operator*

It is usually not possible to give an explicit formula for the whole Lyapunov spectrum, except in some specific cases [21]. Here we propose a mean-field ansatz which gives good results for the slowest modes, and has a nice interpretation in terms of random walk. It is based on the following observations.

The Lyapunov exponents are the eigenvalues of the matrix $\Lambda=E_L[\Lambda]=E_L[\lim_{t\to\infty}(\tilde{D}\mathbf{F}^t_{\mathbf{X}}\cdot D\mathbf{F}^t_{\mathbf{X}})^{1/2t}]$. Since the matrix $\tilde{D}\mathbf{F}^t_{\mathbf{X}}\cdot D\mathbf{F}^t_{\mathbf{X}}$ is bounded in $L_2$ norm, $\forall\mathbf{X}$, $\forall t$, from the Lebesgue theorem one has $\Lambda=\lim_{t\to\infty}E_L[(\tilde{D}\mathbf{F}^t_{\mathbf{X}}\cdot D\mathbf{F}^t_{\mathbf{X}})^{1/2t}]$. On the other hand, the matrix

$$\mathcal{L}(t)=E_L[D\mathbf{F}_{\mathbf{X}}^t] \tag{39}$$

characterizes the (ensemble) average energy transport in $t$ time steps. However, the connection between $\mathcal{L}(t)$ and $\Lambda$ is loose.

Were the transport to be normal, namely, were $D\mathbf{F}_{\mathbf{X}}$ to be independent of $\mathbf{X}$ and of the form $D\mathbf{F}_{\mathbf{X}}=I+\gamma\Delta$, where $\gamma$ is some constant, then $\mathcal{L}(t)$ would be equal to $(I+\gamma\Delta)^t$. In this case, $\mathcal{L}(t)$ would be constant and symmetric. Then $E_L[(\tilde{D}\mathbf{F}_{\mathbf{X}}^t\cdot D\mathbf{F}_{\mathbf{X}}^t)^{1/2t}]=I+\gamma\Delta$ and $\Lambda=I+\gamma\Delta$. In this case the Lyapunov exponents would be eigenvalues of a one-step transport operator $\mathcal{L}=I+\gamma\Delta$ (Fourier modes).

More generally, the (naive) hope would be that of finding an effective transport operator $\mathcal{L}$ such that $\mathcal{L}(t)=\mathcal{L}^t$, and whose singular values (or eigenvalues if $\mathcal{L}$ is self-adjoined) would give the Lyapunov spectrum. However, there is *a priori* no hope of finding such an operator, in general. Note, in particular, that the assumption of independence of the matrices $D\mathbf{F}_{\mathbf{X}(t)}$, the first step toward a mean-field approach, is not a sufficient condition. Since, in this case, $E_L[D\mathbf{F}_{\mathbf{X}}^t]=E_L[D\mathbf{F}_{\mathbf{X}}]^t$, one is led to propose $\mathcal{L}=E_L[D\mathbf{F}_{\mathbf{X}}]$ as a one-step operator. However, one needs further conditions to insure that the singular values of $\mathcal{L}$ give the Lyapunov exponents (see, for example, Refs. [21,22]). It appears, nevertheless, that in Zhang's model an effective transport operator can be found from a mean-field treatment which well approximates the *slowest modes*.

The first obstacle toward a mean-field approach lies in the independence assumption. The matrix $\mathcal{L}(t)$ is a sum of time correlations terms of the form

$$E_L[S(\mathbf{X}(t_{r-1}))S(\mathbf{X}(t_{r-2}))\cdots S(\mathbf{X}(t_0))]$$

whose entry $(i,j)$ writes

$$\Sigma_{i_1,\ldots,i_{r-1}}\Delta_{i,i_{r-1}}\ldots\Delta_{i_2,i_1}\Delta_{i_1,j}\,\mathrm{Prob}[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1,\ldots,Z_{i_1}(\mathbf{X}(t_1))=1,Z_j(\mathbf{X}(t_0))=1]$$

.

Clearly, the nonvanishing terms in this sum are those corresponding to a path from $j$ to $i$, where each intermediate site has been active at least once with a nonzero probability. A simple glance at this formula shows that *a priori all time correlation functions of the joint probability of active sites*, $\mathrm{Prob}[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1,\ldots,Z_{i_1}(\mathbf{X}(t_1))=1,Z_j(\mathbf{X}(t_0))=1]$ have to be considered.

However, Zhang's model, as a hyperbolic dynamical system, has an exponential correlation decay (for finite $L$). The largest correlation decay rate is given by the entropy $\overline{\omega_L}\log(N)$. This decay rate is quite a bit faster than the characteristic times related to the slow modes (for example, the correlation decay rate of a site with itself is about $-0.025$ for $E_c=2.2$, $\epsilon=0.1$, and $L=20$, corresponding roughly to the 320th exponent in the spectrum, while the slowest Lyapunov exponent value is $-0.000\,209\,871$). More generally, we show in Sec. IV that $\lambda_L(1)\sim\overline{\omega_L}/L^d$, to be compared to the decay rate $\overline{\omega_L}\log(N)$. On the other hand, for slow modes, a small perturbation has essentially no variation during one step of an avalanche. In other words, the slowest Oseledec modes are not sensitive to the fast changes (one local time step) of the individual matrices $D\mathbf{F}_{\mathbf{X}}^t$ [the fluctuations of the $Z_j(\mathbf{X}(t))$'s] but, rather, to average variations on the characteristic time scale $t_L(i)=1/\lambda_L(i)$, which is quite a bit longer than a local time step. This suggests that one should consider the projection of the matrices $D\mathbf{F}_{\mathbf{X}(t)}$'s on the slow Oseledec space as independent. This leads to propose $E_L[D\mathbf{F}_{\mathbf{X}}]=I+\alpha\Delta\rho_L I$ as a one-step transport operator. Note that we obtain the same result by assuming that $Z_i(\mathbf{X}(t))$'s are *independent*. Indeed, in this case

$$\mathrm{Prob}[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1,\ldots,Z_{i_1}(\mathbf{X}(t_1))=1,Z_j(\mathbf{X}(t_0))=1]=\rho_L(i_{r-1})\ldots\rho_L(i_1)\rho_L(j)$$

.

Then $\mathcal{L}(t)=\Sigma_{k=1}^t C_t^k(\Delta\rho_L I)^k=(I+\alpha\Delta\rho_L I)^t$.

This approximation gives correct results, provided that one multiplies the density of active sites by 2. This approximation neglects an important effect. Provided $E_c>\epsilon/(1-\epsilon)$, a site *cannot relax for two successive time steps* [11], and therefore it relaxes at most only half of the time during one avalanche. This means, in particular, that the random variables $Z_i(\mathbf{X}(t)),Z_i(\mathbf{X}(t+1))$ are *not independent*, and that the probability that one site relaxes at time $t+1$ depends on what happened at time $t$. In addition, two neighboring sites cannot be simultaneously active. In a certain sense, the lattice is ''blinking:'' during one avalanche all active sites are at pairwise distance [11]. This, therefore, introduces strong correlations between $\mathbf{Z}(\mathbf{X}(t))$ and $\mathbf{Z}(\mathbf{X}(t+1))$.

However, one can circumvent the problem by reparametrizing the time and considering the evolution of the process every *two times steps*. Equivalently, one replaces the stochastic process $\{\mathbf{Z}(\mathbf{X}(t))\}_{t=1}^{+\infty}$ by a new process $\{\mathbf{Y}(t')\}_{t'=1}^{+\infty}=\{\mathbf{Z}(\mathbf{X}(t)),\mathbf{Z}(\mathbf{X}(t+1))\}_{t=1}^{+\infty}$, whose components $Y_k(t)$ take values in $\{0,1\}^2$, where the event (1,1) has zero probability and where $t'=t/2$. One can then encode the $Y_k(t')$ values by $0\rightarrow(0,0)$ (no relaxation at times $t,t+1$) and $1\rightarrow(0,1),(1,0)$ (relaxation at time $t$ or at time $t+1$). This leads to the definition of a new ''density of active sites'' $\rho_L'(i)=\mathrm{Prob}[Y_i(t')=1]=\mathrm{Prob}[Z_i(\mathbf{X}(t))=1$ or $Z_i(\mathbf{X}(t+1))=1]$. Since the events $\{Z_i(\mathbf{X}(t)=1)\}$ and $\{Z_i(\mathbf{X}(t)+1)=1\}$ are disjoints, we have $\rho_L'(i)=\mathrm{Prob}[\mathbf{Z}(\mathbf{X}(t))=1]+\mathrm{Prob}[\mathbf{Z}(\mathbf{X}(t+1))=1]=2\rho_L(i)$. Assuming now that the $Y_k(t')$'s are independent, and considering $\rho_L'(i)$ as the effective density of active sites, one obtains an effective transport operator:

$$\mathcal{L}=I+2\alpha\Delta\rho_L I \tag{40}$$

Calling $\gamma_i$ the singular values of $\mathcal{L}$, our mean-field ansatz suggests that the slowest Lyapunov modes are given by

$$\lambda_L(i)=\log(\gamma_i). \tag{41}$$

Note that this operator is self-adjoined for the metric $\rho_L I$, and that the corresponding matrix can be made symmetric by the variable change $\rho_L^{-1/2}I$.

To check the validity of this ansatz, we first computed the density of active sites on a $20\times20$ lattice, and numerically

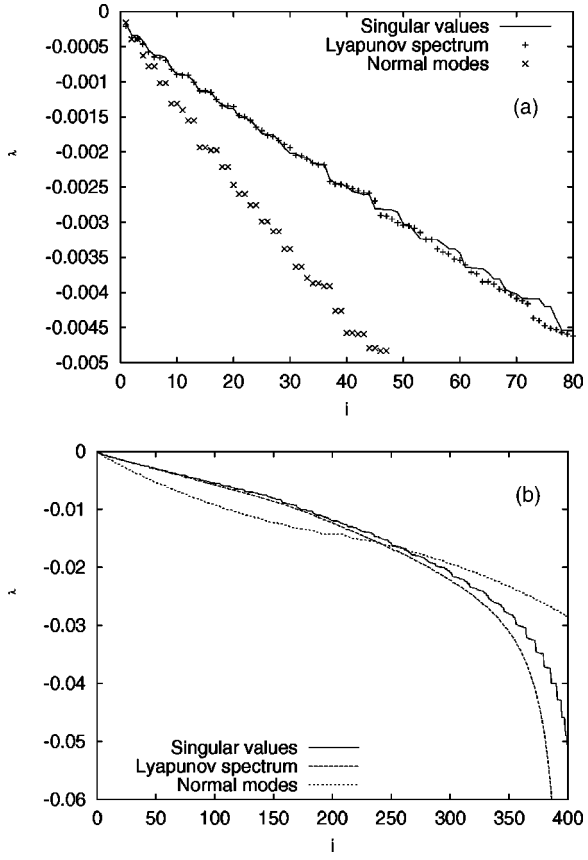


FIG. 2. Lyapunov spectrum, logarithm of the $\mathcal{L}$ singular values, and normal diffusion modes for $E_c=2.2$, $\epsilon=0.1$, and $L=20$. (a) 80 first modes. (b) Full spectra drawn with lines in order to see the shape better.

found the $\gamma_i$'s from these data.[6] At the same time we computed the Lyapunov spectrum. A plot of the two curves is drawn in Fig. 2. One finds a very good agreement over a large part of the spectrum, and the discrepancy increases toward small times scale, as expected.

#### *4. Role of the spatial variations in the density of active sites*

It is usually assumed in the SOC literature, when dealing with the model's spatial properties, that only the density of active sites, and more precisely, its lattice average $\rho_L^{av}$, has to be taken into account. One therefore neglects the spatial dependence of $\rho_L$. In our approach this would lead to an effective transport operator $I+2\alpha\rho_L^{av}\Delta$, corresponding to normal transport. In this case, the slowest Oseledec modes would simply be the eigenmodes of the Laplacian with zero boundaries conditions on $^{\partial}\Lambda$, and the Lyapunov exponents would then correspond to the normal diffusion modes

$$\lambda_L(i)=\log\left\{\left|1+2\alpha\rho_L^{av}\sum_{k=1}^{d}\left[\cos\left(\frac{\pi n_k}{L+1}\right)-1\right]\right|\right\}, \quad (42)$$

where the Laplacian modes are parametrized by the quantum numbers $\mathbf{n}=(n_k)$, $k=1,\ldots,d$, sorted such that the corresponding eigenvalues are decreasing (and $i$ refers to the placement of the exponent in this sequence). In Fig. 2, we also plot the diffusion eigenvalues of Eq. (42). The computed Lyapunov exponents are different from these values except for the largest ones. This approximation is therefore too crude, and gives a wrong spectrum. Note in particular that the *shape* of the spectrum differs, namely, the discrepancy cannot be corrected by a mere multiplication of $\rho_L^{av}$ by some factor. Since the Lyapunov exponents contain all the relevant information about the dynamics at stationarity, our conclusion is that the nonhomogeneity of $\rho_L(i)$ plays a key role in computing dynamical quantities, and implies, unfortunately, that the zeroth order ''mean-field'' approaches, which approximate the density of active sites as a constant, lead to incorrect estimates for finite size when dealing with intermediate time scales. On the other hand, this should lead to correct results when dealing with the longest time scales, since the first modes are well approximated by a transport operator where $\rho_L(i)$ is considered as uniform.

In the literature one often encounters a (apparent) contradiction (see, for example, the original paper by Zhang [5] and a subsequent analysis by Pietronero *et al.* [24]). One assumes the transfer of energy on large time scales to be normal diffusion, while at the same time an anomalous diffusion exponent $z\neq 2$ is computed. It was certainly clear in the spirit of these authors that one has to distinguish the average transport on many avalanches (long time scales) from the average transport within one avalanche (characterized by $z$). Our results on the Lyapunov spectrum makes this distinction quantitative. We show that transport on the longest time scales is normal with a good approximation, while the average transport within *one* avalanche, on time scales corresponding roughly to the crossover point $\lambda_L(i_c)$, is clearly anomalous.

#### *5. Random walk picture*

The operator $\mathcal{L}$ is the Laplace-Beltrami operator associated with random diffusion in a ''medium'' or a landscape *that is not flat*, corresponding to the metric $g=\rho_L I$, where $I$ is the identity matrix on $\mathbb{R}^N$. This has a nice interpretation in the so-called random walk picture.[7] Assume, for a moment, that the energy of a site is composed by (undivisible) energy quanta $\eta$ that can be made arbitrarily small (this is way to ''map'' Zhang's model to a sandpile). Assume that we are in the stationary regime, and that at an initial time we drop a grain in some place and study its motion. At each time step where it is involved in a relaxation process, this grain makes

[6]We were not able to go beyond $L=20$ in the Lyapunov spectrum computation. We used a version of the Eckmann-Ruelle algorithm [17] revisited by Von Bremmen *et al.* [23]. Nevertheless, we needed two weeks of computation on a Pentium II 300 for the case $L=20$, with a relative accuracy of $10^{-3}$.

[7]B.C. is very grateful to P. Grassberger and D. Dhar for illuminating discussions on this point in Trieste.

a jump at random in one of the $2d$ directions in the lattice. From this point of view, the stochastic dynamics of the grain is driven by the underlying dynamics of Eq. (1). *If we assume that the evolution is Markovian*, the probability of jumping from $i$ to some nearest neighbor $j$ depends only on the state of $i$ at time $t$, and is given by a transition rate $W_{ij}=\alpha\rho_L(i)$, while the probability of staying at the same place is $1-\rho_L(i)(1-\epsilon)$ [remember that only an amount $\alpha=(1-\epsilon)/2d$ of the energy is transferred to the neighbors when a site relaxes]. From this consideration, one obtains the equation for the probability $P(i,t)$ that a grain is at place $i$ at time $t$ before it leaves the lattice $P(\cdot,t+1)=[I+\alpha\Delta\rho_L]P(\cdot,t)=[I+\alpha\Delta\rho_L]^tP(\cdot,1)$, and one recovers the operator obtained above when assuming that the $Z_i(t)$'s were independent. Indeed, the independence assumption of the $Z_i(t)$'s is *equivalent to the Markovian assumption in the random walk picture*. The probability of jumping for a grain at time $t$ depends *a priori* on its whole past via a Chapman-Kolmogorov equation, whose transfer matrix is a sum of terms containing conditional probabilities

$$\begin{aligned}&\mathrm{Prob}[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1|Z_{i_{r-2}}(\mathbf{X}(t_{r-2}))=1,\ldots,Z_{i_1}(\mathbf{X}(t_1))=1,Z_j(\mathbf{X}(t_0))=1]\\&=\frac{\mathrm{Prob}[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1,Z_{i_{r-2}}(\mathbf{X}(t_{r-2}))=1,\ldots,Z_{i_1}(\mathbf{X}(t_1))=1,Z_j(\mathbf{X}(t_0))=1]}{\mathrm{Prob}[Z_{i_{r-2}}(\mathbf{X}(t_{r-2}))=1,\ldots,Z_{i_1}(\mathbf{X}(t_1))=1,Z_j(\mathbf{X}(t_0))=1]}\\&=\mathrm{Prob}[Z_{i_{r-1}}(\mathbf{X}(t_{r-1}))=1],\end{aligned}$$

where the last equality holds when the $Z_k(t)$'s are independent. In this case, $W_{ij}=\alpha\,\mathrm{Prob}[Z_i(\mathbf{X}(t))=1]=\alpha\rho_L(i)$ for the $j$ nearest neighbors of $i$.

However, we saw above that the process is not strictly Markovian, since a jump from a given site cannot take place at two successive time steps. In other words, the probability of a jump $i\to j$ depends on the state of $i$ at time $t$ *and* at time $t-1$. The system has some memory (at least two time steps). However, defining the random variables $Y_k(t)$'s, as above, and assuming them to be independent, amounts to rendering the random walk Markovian by a suitable reparametrization of the process, and gives a transfer equation

$$P(\cdot,t+1)=[I+2\alpha\Delta\rho_L]P(\cdot,t)=\mathcal{L}^tP(\cdot,1) \tag{43}$$

Therefore, the operator $\mathcal{L}$ characterizing the decay of a small perturbation can also be interpreted as the transfer matrix of a random walk, in a medium where the diffusion rate depends on the location.

#### *6. Density of active sites and average energy flowing towards the boundaries*

Equation (43) characterizes the energy transport in the lattice, but does not take into account the source term (addition of a grain) required to reach stationarity. Indeed, each time a grain exits the lattice, one must add another grain at a random place $i$, with probability $\overline{\omega_L}(i)$ ; this is a source term. Call $\mathcal{P}_L$ the equilibrium state of the random walk, and $\mathcal{V}(\partial\lambda)$ the set of sites at distance 1 from the boundary. The probability for a grain to exit is $2\alpha\Sigma_{j\in\mathcal{V}(\partial\lambda)}\rho_L(i)\mathcal{P}_L(i)$. This is obviously proportional to the outgoing energy flux, which is, at stationarity, equal to the incoming flux (the probability of adding a grain in the lattice), namely[8]

$$\overline{\omega_L}=2\alpha\sum_{j\in\mathcal{V}(\partial\lambda)}\rho_L(j)\overline{X_L}(j). \tag{44}$$

The complete equation for the energy at stationarity is

$$2\alpha\Delta[\rho_L\overline{\mathbf{X}}_L]+\overline{\omega_L}(i)=0, \tag{45}$$

with zero boundaries conditions and with constraint (44).

In this equation one distinguishes a *local* transport term, and a source term which depends on a *global* constraint. When the excitation is uniform, Eq. (45) reduces to $\Delta[\rho_L\overline{\mathbf{X}}_L]+(\overline{\omega_L}/2\alpha L^d)=0$.

The difficulty in solving this equation is that it deals with the product $\rho_L\overline{\mathbf{X}}_L$. On the other hand, it is known in the literature that $\overline{\mathbf{X}}_L$ converges to a uniform energy distribution over the lattice as $L\to\infty$ [25]. Assume now that we can write $\overline{\mathbf{X}}_L$ as

$$\overline{\mathbf{X}}_L=\overline{\mathbf{X}}_0+f(L), \tag{46}$$

where $\|f(L)\|$ goes to zero as $L\to\infty$, and where $\overline{\mathbf{X}}_0$ is spatially uniform, i.e., $\overline{\mathbf{X}}_0(i)=\mathrm{const}=\overline{x_L}$. At zeroth order, for $\rho_L$ one obtains the equation

[8]The cautious reader has noted that this equation is not dimensionally correct, since no energy term appears on the left-hand side. One should indeed multiply the left-hand side by $\delta$, the input energy quantum, which is set to 1 throughout this paper.

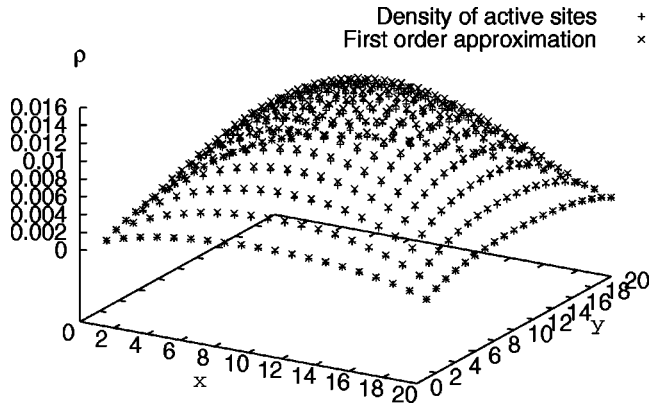


FIG. 3. Plot of the density of active sites and solution of Eq. (47) for $E_c=2.2$, $\epsilon=0.1$, and $L=20$.

$$\Delta\rho_L+\frac{\overline{\omega_L}}{2\alpha L^d\overline{x_L}}=0, \tag{47}$$

where $L^d\overline{x_L}=E_{tot}$, the average total energy in the lattice. The solution of this equation can easily be found by decomposition on the eigenmodes of the Laplacian. The general solution is

$$\rho_L(\mathbf{x})=\sum_{\mathbf{n}} A_{\mathbf{n}}\prod_{i=1}^{d}\sin(k_i x_i), \tag{48}$$

where $\mathbf{n}=(n_1,\ldots,n_d)$ is the set of quantum numbers parametrizing the eigenmodes of the discrete Laplace operator, $s_{\mathbf{n}}=2[\Sigma_{i=1}^{d}\cos(k_i)-d]$ is the corresponding eigenvalue with $k_i=n_i\pi/(L+1)$,

$$A_{\mathbf{n}}=-\frac{2^{d-1}\overline{\omega_L}}{\alpha E_{tot}(L+1)^d}\frac{\prod_{i=1}^{d}C_{n_i}}{s_{\mathbf{n}}},$$

and

$$C_{n_i}=\sum_{x=1}^{L}\sin(k_i\cdot x)=(-1)^{m_i}\frac{\sin\left(\frac{n_i\pi L}{2(L+1)}\right)}{\sin\left(\frac{n_i\pi}{2(L+1)}\right)},$$

where $n_i=2m_i+1$. Surprisingly, this already gives quite a good approximation for $\rho_L$, which becomes better and better as $L$ increases (see Figs. 3 and 4).

Away from the boundaries, one expects rotational invariance for $\rho_L(\mathbf{x})$. This can be checked by expanding the function sin near to $x_i=L/2$, $i=1,\ldots,d$ up to third order. One obtains the well known paraboloid form [26] $\rho_L(\mathbf{x})\sim K_0-K_1\Sigma_{i=1}^{d}x_i^2$, where the constants $K_0$ and $K_1$ can be easily deduced from Eq. (47).

One also obtains the average density of active sites, $\rho_L^{av}=(1/L^d)\Sigma_{i=1}^{N}\rho_L(i)$,

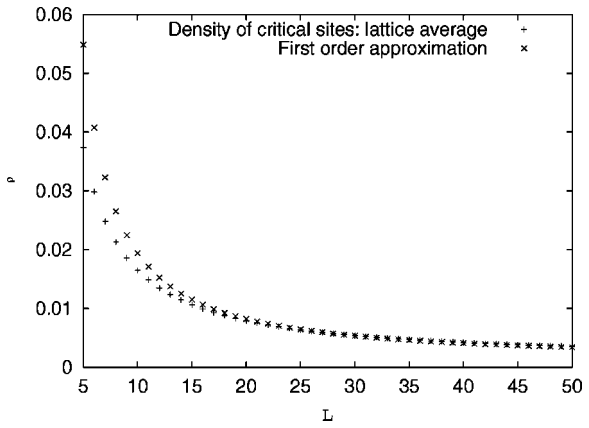


FIG. 4. Plot of $\rho_L^{av}$ and solution of Eq. (49) vs $L$, for $E_c=2.2$ and $\epsilon=0.1$.

$$\rho_L^{av}=-\frac{2^{d-1}\cdot\overline{\omega_L}}{L^d(L+1)^d\alpha E_{tot}}\sum_{\mathbf{n}}\frac{\prod_{i=1}^{d}C_{n_i}^2}{s_{\mathbf{n}}}, \tag{49}$$

which is expected to hold for sufficiently large $L$. In Fig. 4 we show a plot in which it clearly appears that this formula gives already a quite good estimate for $L=15$.

## IV. SCALING PROPERTIES OF THE LYAPUNOV SPECTRUM

Zhang's model, as a hyperbolic dynamical system, cannot exhibit a critical behavior for finite size, since it has an exponential correlation decay.[9] However, since a critical behavior is conjectured in the thermodynamic limit, one expects that hyperbolicity is lost as $L\to\infty$, namely, some of the Lyapunov exponents go to zero. It is therefore of crucial importance to know the behavior of the Lyapunov exponents as $L\to\infty$. In this section, we first discuss the time scale separation between the activation rate and the dissipation rate, which is believed to be a fundamental ingredient to have SOC, and its links to the Lyapunov exponents. We then show that using a finite size scaling ansatz provides a scaling exponent from which the scaling of some SOC observables can be obtained.

### A. Time scale separation

Equation (49) can be written as

$$\rho_L^{av}=-\frac{2^{d-1}\overline{\omega_L}}{L^d\alpha E_{tot}}\gamma_L, \tag{50}$$

[9]The exponential correlation decay is a general property of hyperbolic systems, but in the presence of singularities one can also observe a polynomial correlation decay and a weak initial condition senstivity [9].

where $(L+1)^d\gamma_L=\Sigma_{\mathbf{n}}(\Pi_{i=1}^d C_{n_i}^2/s_{\mathbf{n}})$. Let us estimate the scaling of this sum as $L\to\infty$. First, set $d=1$ and fix $\alpha>0$ arbitrarily small. The sum over $n=n_1$ can be split into one part such that $n<(L+1)\alpha$ and another part where $n\geqslant(L+1)\alpha$. In the first sum, $C_n\sim 2(L+1)/n\pi$ and $s_{\mathbf{n}}\sim -(\pi n)^2/(L+1)^2$, while the second sum is smaller than $(L+1)C(\alpha)$, where $C(\alpha)$ is bounded for $\alpha>0$. Therefore $\gamma_L\sim(L+1)^3S$, where $S\sim\Sigma_{n<(L+1)\alpha}(1/n^4)$ remains bounded as $L\to\infty$. Then $\gamma_L\sim(L+1)^3$. This argument can be generalized for any $d$ by splitting the sum over $\mathbf{n}=(n_1,n_2,\ldots,n_d)$ into sums where $k$ indexes are smaller that $\alpha(L+1)$, $k$ going from 0 to $d$. It is easy to see that the main contribution is due to the terms such that $d$ indexes are $<\alpha(L+1)$, giving a leading contribution $\Sigma(L+1)^{2d+2}$ and $\gamma_L\sim L^{d+2}$. We therefore conclude that $\rho_L^{av}$ scales like

$$\rho_L^{av}\sim\frac{\overline{\omega}_L}{E_{tot}}\frac{(L+1)^{d+2}}{L^d}\sim\frac{\overline{\omega}_L L^2}{E_{tot}}=\frac{\overline{\omega}_L L^2}{L^d\overline{X}_0}. \tag{51}$$

We set $h=\overline{\omega}_L/L^d$ for the driving rate, and assume that $e=h/\rho_L^{av}=\overline{x}_L L^{-2}$. One obtains the energy conservation equation $h=\rho_L^{av}e$, and therefore $e$ is the energy dissipated per active site and per unit time. This corresponds to the *dissipation rate* introduced by Vespignani *et al.* [15]. Since $0<\overline{x}_L<E_c$, $\forall L$, $\overline{x}_L$ plays no role in the asymptotic scalings in $L$, and therefore $e\sim L^{-2}$, as already anticipated by a mean-field approach in Ref. [15].

The average value of observables like size, duration, etc. is known to diverge with a power law scaling $\langle x\rangle_L\sim L^{\gamma_x}$. Therefore $1/\langle\tau\rangle_L\to 0$ like $L^{-\gamma_\tau}$ as $L\to\infty$ where $\gamma_\tau>1$ [3]. Since $0\leqslant\overline{\omega}_L\leqslant 1$ [see Eq. (20)], Eq. (21) implies that $p_L=f(1/\langle\tau\rangle_L)=(a_1/\langle\tau\rangle_L)-(a_2/\langle\tau\rangle_L^2)+O(1/\langle\tau\rangle_L^3)$. This is particularly clear for $E_c<1$, since $p_L=\overline{\omega}_L$, which implies $p_L=1/(1+\langle\tau\rangle_L)$, and therefore $a_1=1,a_2=1$. For general $E_c$, using this form gives, from Eq. (21), $a_1=1$,

$$p_L\sim\frac{1}{\langle\tau\rangle_L}-\frac{a_2}{\langle\tau\rangle_L^2}, \tag{52}$$

and

$$\overline{\omega}_L\sim\frac{a_2}{\langle\tau\rangle_L} \tag{53}$$

as $L\to\infty$. It follows, therefore, that

$$\rho_L^{av}\sim L^{-\gamma_\tau+2-d} \tag{54}$$

We have therefore shown that

$$h\to 0,\quad e\to 0,\quad \rho_L^{av}=\frac{h}{e}\to 0\quad\text{as}\quad L\to\infty \tag{55}$$

In Ref. [15] Vespignani *et al.* discussed the necessity of this triple limit in order to have SOC. However, in their analysis the activation and dissipation rate were free parameters (tunable ''by hand''). In Zhang's model, $h$ and $e$ are not free, since they are fully determined by the dynamics. Therefore, we have shown that the three limits discussed by Vespignani *et al.* [15] are indeed achieved, without external fine tuning of some parameter, in Zhang's model, by the simple constraints one imposes on the dynamics (adiabatic driving).

From Eq. (53) we have that the positive Lyapunov exponent (the entropy) $\overline{\omega}_L\log(N)\to 0$ in the thermodynamic limit. On the other hand, the first negative Lyapunov exponent is given with a good accuracy by the normal diffusion operator $1+2\rho_L^{av}\Delta$ (see Fig. 2), which implies that $\lambda_L(1)\sim\rho_L^{av}L^{-2}$. Another way of arguing is to note that from theorem 1, $\lambda_L(1)$ scales like the average ratio of energy dissipated by one site. From the local conservation of energy, $\lambda_L(1)\overline{x}_L\sim h$; then $\lambda_L(1)\sim\rho_L^{av}L^{-2}$. Therefore, $\lambda_L(1)\to 0$ in the thermodynamic limit. Actually, the finite-size scaling analysis of Sec. IV B suggests that a large number of negative Lyapunov exponents also go to zero as $L\to\infty$. But the double limit $\lambda_L(0)\to 0,\lambda_L(1)\to 0$ already shows that the *hyperbolicity is lost in the thermodynamic limit*. Note, however, that these two exponents are not independent, since local conservation of energy imposes $\lambda_L(1)\overline{x}_L\sim h$ which implies $\lambda_L(1)/\lambda_L(0)\sim L^{-d}/\log(N)$. Incidentally, this validates the separation of time scale between the correlation decay time $1/\lambda_L(0)$ and the largest transport characteristic time $1/\lambda_L(1)$ we used in Sec. III when deriving the mean-field transport equation for the slowest modes.

### B. Finite size scaling of the Lyapunov spectrum

An approximate expression for modes related to transport in the lattice is obtained from the operator $\mathcal{L}$ [Eq. (40)], whereas an approximate equation for $\rho_L$ is given by Eq. (49). However, at the moment we do not have an analytical expression for the modes of $\mathcal{L}$. In this sequel, we restrict to the scaling of the slowest singular values of $\mathcal{L}$ with the system size.

When dealing with a scaling analysis in the thermodynamic limit, one usually first tries to use finite size scaling (FSS). This is a standard tool in statistical mechanics. It has also been proposed in SOC as an ansatz for the scaling of the probability distribution of avalanches observables [7]. However, its validity was recently questioned in this case [8].

Nevertheless, since this is certainly the first ansatz one can use to try to do a scaling analysis, in this section we attempt a finite-size scaling ansatz for the Lyapunov spectrum, and look at the results and conclusions we are led to.[10] We assume, therefore, that, for any $L$, there exists a change of coordinates $i\to\phi_L(i),\lambda_L\to\psi_L(\lambda_L)$, depending on $L$, such that the points of the spectrum $\{i,\lambda_L(i)\}$ are mapped onto the same ''universal'' curve[11] $\{x,\lambda(x)\}$, where $\lambda(x)=\psi_L\circ\lambda_L\circ\phi_L^{-1}(x)$. Furthermore we assume (as in usual finite size scaling) that the coordinate changes are simple dilata-

[10]Note that the FSS of the Lyapunov spectrum is not a general property of dynamical systems, even close to a phase transition point [27,28].

[11]Note that this curve depends on the parameters $E_c$, $\epsilon$, and $d$.

TABLE I. Computed values of $\tau_\lambda$ vs $E_c$, obtained from Eq. (59), for samples of size $L=10$–$20$.

| $E_c$ | $\tau_\lambda$ |
|---|---|
| 0.6 | 0.632 |
| 1.1 | 0.622 |
| 1.5 | 0.621 |
| 2.2 | 0.560 |
| 4.1 | 0.524 |

tions where $\phi_L(x)=L^{\beta_\lambda}\cdot x$, $\psi_L(x)=L^{\beta_\lambda\cdot\tau_\lambda}(x)$. Then

$$\lambda(x)=L^{\beta_\lambda\tau_\lambda}\lambda_L(xL^{-\beta_\lambda}). \tag{56}$$

Equivalently, knowing the curve $\{x,\lambda(x)\}$, the spectrum for a given size is

$$\lambda_L(i)=L^{-\beta_\lambda\tau_\lambda}\lambda(iL^{-\beta_\lambda}),\quad i=1,\ldots,L^d. \tag{57}$$

Since the set of indices $i\in\{1,\ldots,L^d\}$ it is evident that

$$\beta_\lambda=d. \tag{58}$$

The exponent $\tau_\lambda$ can be numerically computed by several means. The first is to minimize the Euclidean distance between the spectra obtained for different lattice sizes, with respect to $\tau_\lambda$. Another way is to compute the sum of the Lyapunov exponents. Indeed

$$S_L\overset{\text{def}}{=}\sum_{i=1}^{N}\lambda_L(i)=L^{-d\tau_\lambda}\sum_{y=L^{-d}}^{1}\lambda(y)\sim L^{d(1-\tau_\lambda)}\int_{L^{-d}}^{1}\lambda(y)dy.$$

Assuming that $\lambda(y)$ is bounded as $y\to 0$, and that $0<K=\int_0^1\lambda(y)dy<\infty$, one obtains

$$S_L\sim K\cdot L^{d\cdot(1-\tau_\lambda)}, \tag{59}$$

which allows one to compute $\tau_\lambda$. The value of $\tau_\lambda$ for $d=2$, and $\epsilon=0.1$ and different $E_c$ values are given in Table I. These values were obtained for a sample of spectra from $L=10$ to 20. We note, in particular, that $\tau_\lambda$ depends on $E_c$. At the moment we have no way of knowing whether this effect persists in the thermodynamic limit. Note that these values are given as indications, but that a correct estimation of $\tau_\lambda$ certainly requires further investigations for consequently larger system sizes. These numerical studies are beyond our present computer performances.

The data collapse of spectra is drawn in Fig. 5. Though a good data collapse is not sufficient to ensure FSS, Fig. 5 indicates that this gives a good approximation of the spectrum. Actually, we do not expect FSS to hold for the whole spectrum (in particular the kernel modes could have different scalings). For the following discussion, however, it is sufficient to assume that FSS holds for the slowest modes. This is a reasonable assumption, since these modes are well approximated by normal diffusion.

We now relate the exponents $\gamma_s$, and $\gamma_\tau$ (and other characteristic exponents like $z$, the anomalous diffusion exponents) to $\tau_\lambda$. Note that $\gamma_x$ is related to the critical exponent

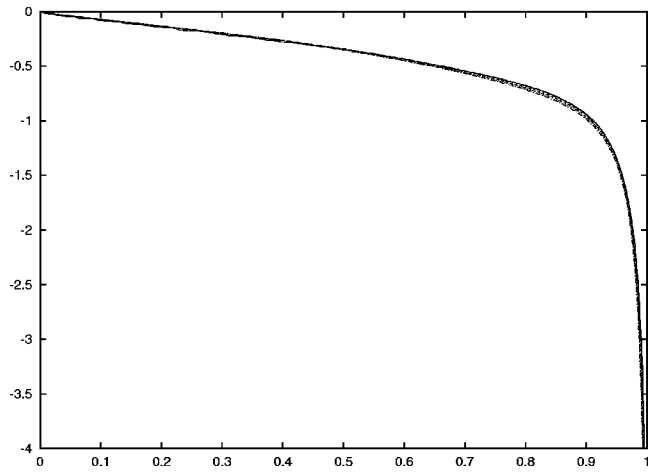


FIG. 5. Data collapse of the Lyapunov spectrum for $E_c=2.2$, $\epsilon=0.1$, and $L=12$, 14, 16, and 18.

$\tau_x$,[12] and therefore our discussion suggests that there is a link between $\tau_\lambda$ and the critical exponents $\tau_s$ and $\tau_\tau$.

FSS leads to $\lambda_L(1)=L^{-d\cdot\tau_\lambda}\lambda(L^{-d})$. However the analyticity properties of $\lambda$ near zero are not known. Assume that $\lambda(x)\sim x^\alpha$ and $x\sim 0$, where $\alpha$ may depend on $d$ (seemingly $\alpha=1$ for $d=2$). Then $\lambda_L(1)\sim L^{-d\tau_\lambda-d\alpha}$. From $\lambda_L(1)\sim\rho_L^{av}L^{-2}$, one obtains

$$\rho_L^{av}\sim L^{-d\tau_\lambda+2-d\alpha}, \tag{60}$$

and, from Eq. (54), $\gamma_\tau+d=d(\tau_\lambda+\alpha)$. Finally, from Eqs. (31), (52), and (59), one obtains:

$$d\tau_\lambda=d-\gamma_s+\gamma_\tau, \tag{61}$$

which gives

$$\gamma_s=2, \tag{62}$$

$$\gamma_\tau=d\tau_\lambda+2-d, \tag{63}$$

and

$$\alpha=\frac{2}{d}. \tag{64}$$

The equation for $\gamma_s$ was already anticipated by many authors on the basis of numerical simulations [25], and the mean-field approach [15] and was proved in Dhar's model for $d=2$ by Dhar himself [29]. The equation for $\alpha$ is well verified at $d=1$ and 2. However, this relation deserves further investigation in larger dimensions. It suggests, in particular, that the curve $\lambda(x)$ is not $C^1$ at zero for $d>2$, i.e., the largest exponents do not go to zero in a smooth way as $L\to\infty$.

Finally, the anomalous diffusion exponent $z$, characterizing the average transport within one avalanche, is equal to $\gamma_\tau$

[12]Under the finite size scaling assumption of $P_L(x)$, one finds that $\gamma_x=\beta_x(2-\tau_x)$, where $L^{\beta_x}$ is the scaling for the maximal value of $x$ in a lattice of size $L$.

if one assumes that the average avalanche radius scales like $L$ in any dimension [3]. Equivalently, one can note that the crossover point for the $\chi_L(i)$ spectrum [Eq. (32)] is $\sim L^z/\langle\tau\rangle_L$ and does not depend on $L$. From Eq. (63) it follows that the transport on time scales of order $\langle\tau\rangle_L$ is anomalous ($z<2$) if $\tau_\lambda<1$. Note, however, that this argument assumes that FSS is still valid at the crossover point. This result suggests therefore that some of the critical exponents of SOC can be obtained from a simple scaling ansatz on the Lyapunov spectrum.

As a final remark, note that the $E_c$ dependence appearing in Table I would have to be clarified, since it suggests that the critical exponents depend on $E_c$. This was already argued in Refs. [9–11], and suggested from numerical simulations (though not discussed) in Ref. [30]. Note, however, that the dependence of dynamical quantities in the control parameter in a dynamical system is more a rule than an exception. One certainly needs very special properties to ensure that the critical exponents are constant in the limit $L\to\infty$, whatever the value of $E_c$. If this happened to be true, it would mean that Zhang's model is somehow nongeneric, at least from a dynamical system point of view.

## V. CONCLUSION

In this paper, we investigated the dynamics of Zhang's model in terms of the Lyapunov exponents and Oseledec modes. Due to the piecewise affine structure of the model, the Lyapunov exponents, usually related to the local properties of the dynamics (expansion rates, fractal dimensions, entropy), also appear as characteristic rates of energy transport in the system. We showed that the spectrum is roughly divided into two parts: slow modes, corresponding to transport and dissipation; and fast modes, essentially associated with the stability of the attractor. Even if the Oseledec modes are analogs of Fourier modes in normal diffusion, they are not normal, because the density of the active sites is not spatially homogeneous. The slow Oseledec modes correspond, rather, to diffusion in a metric which is not flat and is given by the density of active sites. Only for the slowest mode are the Lyapunov exponents the same as for the largest rate in normal diffusion. This is important, since the slowest mode characterizes the equilibrium properties of the model. This means that the usual mean-field approaches, which replace the density of active sites by its lattice average, are correct if one considers properties related to the longest time scales. Since the critical exponents $\gamma_s$ and $\gamma_\tau$ characterize statistical properties on the largest time scale, they are naturally related to the slowest Lyapunov exponent.

We investigated the scaling properties of the spectrum with respect to the lattice size and found that finite size scaling gives a good approximation. In particular we extracted a critical exponent $\tau_\lambda$ which is related to the usual critical exponents computed in the literature. However, there is clearly a lot more information in the Lyapunov spectra than in the usual critical exponents.

The scaling form also shows that in the thermodynamic limit a part of the spectrum goes to zero, corresponding to translation invariance and zero dissipation. In this way Zhang's model is not hyperbolic in the thermodynamic limit. This limit now has to be studied in more detail, especially as far as the vanishing of correlations is concerned. It may indeed be a way to make a connection between SOC and the usual critical phenomena.

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